Blow-up of solutions for a viscoelastic wave equation with m-Laplacian and delay terms

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Abstract

In this work, we deal with the viscoelastic wave equation with m-Laplacian and delay terms. We study blow-up of solutions for positive initial energy under suitable conditions.

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1 Introduction

In this part, we study the viscoelastic wave equation with m-Laplacian and delay terms

$$\begin{cases} |u_t|^{\rho} u_{tt} - \Delta u - \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right) + \int_0^t g(t-s)\Delta u(s)\,ds & \\ -\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) & \\ = b|u|^{p-2}u, & \text{in } \Omega \times (0,\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. $\rho > 0$, $p > m > 2$, μ_1, b are positive constants, μ_2 is a real number, $\tau > 0$ indicates the time delay, the term $\Delta_m u = \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$ is called m-Laplacian, g is the kernel function satisfies some conditions to be specified later. In a suitable function space, (u_0, u_1, f_0) are the initial data.

Time delay appears in many practical problems such as economic phenomena, thermal, biological, chemical and physical [1].

Our aim is to consider a viscoelastic wave equation with m-Laplacian term ($\operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$) and delay term ($\mu_2 u_t(x,t-\tau)$).

In 1986, Datko et al. [2] indicated that delay is a source of instability. In 2006, Nicaise and Pignotti [3] looked into the wave equation with delay term as following

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0. \tag{1.2}$$

Under the condition $0 < \mu_1 < \mu_2$, they proved the stability result.

In the absence of the m-Laplacian term ($\operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$), the equation (1.1) becomes

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}u. \tag{1.3}$$

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Wu [4], studied the equation (1.3) under suitable conditions. He established the blow up result in a finite time.

Liu [5], studied the following viscoelastic equation

$$|u_t|^{\rho} u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\,\Delta u(s)\,ds = b\,|u|^{p-2} u. \tag{1.4}$$

He proved the blow-up result. Also, the author obtained the decay results for the equation (1.4).

Recently, Kafini and Messaoudi [1], studied the following wave equation

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b\,|u|^{p-2} u, \tag{1.5}$$

with delay term. Under suitable conditions, they proved global nonexistence of the equation (1.5). Moreover, some other authors studied related problems (see [6, 7, 8, 9, 10, 11, 12, 13, 14]). Also, some other authors concerned the numerical analysis for some related problems (see [15, 16, 17]).

In this work, we get the blow-up result for positive initial energy. There is no research, to our best knowledge, related to viscoelastic wave equation with a varying material density ($|u_t|^{\rho}$), m-Laplacian term ($\operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$) and delay term ($\mu_2 u_t(x,t-\tau)$), therefore, our paper improves the previous studies.

The outline of this paper is as follows: In Sect. 2, we give needed assumptions and lemmas. In Sect. 3, we get the blow-up results.

2 Preliminaries

In this part, for stating and proving our result, we give some material. We will use the Lebesgue $L^p(\Omega)$ and Sobolev $W_0^{m,p}(\Omega)$ spaces with their norms $\|\cdot\|_p$ and $\|\cdot\|_{W_0^{m,p}(\Omega)}$.

Lemma 2.1. [18, 19] Let $2 \le p \le \frac{2n}{n-2}$, the inequality

$$\|u\|_p \le c_s \|\nabla u\|_2 \text{ for } u \in H_0^1(\Omega),$$

holds with some positive constants c_s.

Suppose that

$$0 < \rho \le \frac{2}{n-2} \text{ if } n \ge 3 \text{ and } \rho > 0 \text{ if } n = 1, 2, \tag{2.1}$$

and

$$m < p \le \frac{mn}{n-m}, \text{ if } n > m \text{ and } p > m \text{ if } n \le m, \tag{2.2}$$

satisfy for ρ and p.

Related to $g(t)$ kernel function, we suppose that:

(A1) $g : R^+ \to R^+$, and

$$g(0) > 0,\ g'(s) \le 0 \text{ and } 1 - \int_0^{\infty} g(s)\,ds = l > 0, \tag{2.3}$$

satisfies.

Also, in [1], from Lemma 2.2 we get lemma as follows:

Lemma 2.2. Assume that (2.2) holds, such that

$$\|u\|_p^s \leq C\left(\|\nabla u\|_2^2+\|\nabla u\|_m^m+\|u\|_p^p\right),$$

where C is a positive constant, satisfies for any $u \in W_0^{1,m}(\Omega)$ and $m \leq s \leq p$.

Now we introduce, similar to the work of [20], the new function

$$z(x,\kappa,t)=u_t(x,t-\tau\kappa) \;\; x\in\Omega,\;\; \kappa\in(0,1),$$

which gives us

$$\tau z_t(x,\kappa,t)+z_\kappa(x,\kappa,t)=0 \text{ in } \Omega\times(0,1)\times(0,\infty).$$

Hence, problem (1.1) transformes to

$$\begin{cases} |u_t|^\rho u_{tt}-\Delta u-\operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right)+\int_0^t g(t-s)\Delta u(s)\,ds & \\ -\Delta u_{tt}+\mu_1 u_t(x,t)+\mu_2 z(x,1,t) & \\ =b|u|^{p-2}u, & \text{in } \Omega\times(0,\infty), \\ \tau z_t(x,\kappa,t)+z_\kappa(x,\kappa,t)=0, & x\in\Omega,\;\kappa\in(0,1),\; t>0, \\ z(x,0,t)=u_t(x,t), & x\in\Omega,\; t>0, \\ z(x,\kappa,0)=f_0(x,-\tau\kappa), & x\in\Omega, \\ u(x,0)=u_0(x),\; u_t(x,0)=u_1(x), & x\in\Omega, \\ u(x,t)=0, & x\in\partial\Omega,\; t\geq 0. \end{cases} \tag{2.4}$$

Next, by combining the arguments [21, 22], we give the local existence theorem of problem (2.4).

Theorem 2.3. Assume that $\mu_2<\mu_1$, (A1), and (2.1)-(2.2) satisfy. Assume that $u_0, u_1 \in W_0^{1,m}(\Omega)$ and $f_0\in L^2(\Omega\times(0,1))$. Hence, there exists a unique solution (u,z), for $T>0$, satisfies

$$\begin{aligned} u, u_t &\in C\left([0,T);W_0^{1,m}(\Omega)\right), \\ z &\in C\left([0,T);L^2(\Omega\times(0,1))\right). \end{aligned}$$

3 Blow-up

In this part, we get the blow-up result for positive initial energy. Firstly, we define the energy functional of the problem (2.4) as follows

$$\begin{aligned} E(t) = {} & \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{m}\|\nabla u\|_m^m+\frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|^2+\frac{1}{2}(g\circ\nabla u)(t) \\ & +\frac{1}{2}\|\nabla u_t\|^2+\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx-\frac{b}{p}\|u\|_p^p, \end{aligned} \tag{3.1}$$

so that

$$\tau|\mu_2|\leq\xi\leq\tau(2\mu_1-|\mu_2|), \tag{3.2}$$

where ξ is a positive constant and

$$(g\circ\nabla\nu)(t)=\int_0^t g(t-s)\|\nu(s)-\nu(t)\|_2^2\,ds.$$

To get the main result, we give the following assumption on g,

$$\int_0^\infty g(s)\,ds < \frac{1}{1+\frac{1}{\left(p(1-\beta)^2+2\beta(1-\beta)\right)(p-2)}}, \tag{3.3}$$

where $0<\beta<1$ is a fixed number.

Lemma 3.1. $E(t)$ is a nonincreasing function, hence

$$\begin{aligned} E'(t) &\le -\alpha\left(\|u_t\|^2+\int_\Omega z^2(x,1,t)\,dx\right)+\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u\|^2 \\ &\le -\alpha\left(\|u_t\|^2+\int_\Omega z^2(x,1,t)\,dx\right)\le 0,\ \text{for } t\ge 0, \end{aligned}$$

where $\alpha=\min\left\{\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2},\ \frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right\}$, which is positive by (3.2).

Proof. Multiplying the first equation in (2.4) by u_t and integrating over Ω and multiplying the second equation in (2.4) by $\frac{\xi}{\tau}z$ and integrating over $(0,1)\times\Omega$ with respect to κ and x and summing up, we obtain

$$\begin{aligned} \frac{d}{dt}E(t) &\le -\mu_1\|u_t\|^2+\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u\|^2-\mu_2\int_\Omega u_t z(x,1,t)\,dx \\ &\quad -\frac{\xi}{\tau}\int_\Omega\int_0^1 zz_\kappa(x,\kappa,t)\,d\kappa dx. \end{aligned} \tag{3.4}$$

Now, estimating the last two terms of the right-hand side of (3.4), respectively, we get:

$$\left|-\mu_2\int_\Omega u_t z(x,1,t)\,dx\right|\le\frac{|\mu_2|}{2}\left(\int_\Omega u_t^2dx+\int_\Omega z^2(x,1,t)\,dx\right) \tag{3.5}$$

and

$$\left|-\frac{\xi}{\tau}\int_\Omega\int_0^1 zz_\kappa(x,\kappa,t)\,d\kappa dx\right|=\frac{\zeta}{2\tau}\left(\int_\Omega u_t^2dx-\int_\Omega z^2(x,1,t)\,dx\right). \tag{3.6}$$

Substituting (3.5)-(3.6) into (3.4) and from (A1), we get

$$\begin{aligned} \frac{d}{dt}E(t) &\le -c_1\|u_t\|^2-c_2\int_\Omega z^2(x,1,t)\,dx+\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u\|^2 \\ &\le -\alpha\left(\|u_t\|^2+\int_\Omega z^2(x,1,t)\,dx\right)\le 0,\ \forall t\ge 0, \end{aligned}$$

where $c_1=\mu_1-\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2}>0$, $c_2=\frac{\zeta}{2\tau}-\frac{|\mu_2|}{2}>$ and $\alpha=\min\{c_1,c_2\}$. Hence, we completed the proof. Q.E.D.

Now, by (3.1), (2.3) and Lemma 2.1, we get

$$\begin{aligned} E(t) &\ge \frac{1}{2}l\|\nabla u\|^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{bB_1^p}{p}\left(l^{\frac{1}{2}}\|\nabla u\|_2\right)^p \\ &\ge F\left(\sqrt{l\|\nabla u\|_2^2+(g\circ\nabla u)(t)}\right),\ t\ge 0, \end{aligned} \tag{3.7}$$

where $B_1 = \frac{c_s^p}{l^{\frac{p}{2}}}$ and

$$F(x) = \frac{1}{2}x^2 - \frac{bB_1^p}{p}x^p,\ x > 0.$$

Remark 3.2. Similar to [23], we know that, the functional F is increasing in $(0, \lambda_1)$, decreasing in (λ_1,∞), and F has a maximum at $\lambda_1 = b^{-\frac{1}{p-2}} B_1^{-\frac{p}{p-2}}$ with the maximum value $E_1 = F(\lambda_1) = \frac{p-2}{2p} b^{-\frac{2}{p-2}} B_1^{-\frac{2p}{p-2}} = \frac{p-2}{2p}\lambda_1^2$.

Lemma 3.3. [4] Assume that (2.1)-(2.2) and (A1) satisfy and suppose that $l\|\nabla u_0\|^2 > \lambda_1^2$ and $E(0) < E_1$, then there exists $\lambda_2 > \lambda_1$, so that

$$l\|\nabla u\|^2 + (g \circ \nabla u)(t) \geq \lambda_2^2, \tag{3.8}$$

for all $t \in [0, T)$, and

$$\|u\|_p^p \geq \frac{bB_1^p}{p}\lambda_2^p. \tag{3.9}$$

Theorem 3.4. Let (2.1), (2.2), (3.2), (3.3) and (A1) hold. Assume that $u_0,\ u_1 \in W_0^{1,m}(\Omega)$ with $l\|\nabla u_0\|^2 > \lambda_1^2$ and $E(0) < \beta E_1$. Suppose further that $\rho < p-2$. Then, the solution of (2.4) blows up in finite time.

Proof. From contradiction, we assume that the solution of problem (2.4) is global, such that

$$\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|u\|_p^p + \|\nabla u_t\|^2 + \|\nabla u\|^2 \leq K_1,\ \forall t \geq 0, \tag{3.10}$$

where $K_1 > 0$.

We set, $E_2 \in (E(0), \beta E_1)$, such that

$$H(t) = E_2 - E(t).$$

From lemma 3.1, (3.8) and $E_1 = \frac{p-2}{2p}\lambda_1^2$, we get

$$H(t) \geq H(0) = E_2 - E(0) > 0 \tag{3.11}$$

and

$$\begin{aligned} H(t) &\leq \beta E_1 - \frac{1}{2}\left(l\|\nabla u\|^2 + (g \circ \nabla u)(t)\right) + \frac{b}{p}\|u\|_p^p \\ &\leq E_1 - \frac{1}{2}\lambda_1^2 + \frac{b}{p}\|u\|_p^p \leq \frac{b}{p}\|u\|_p^p. \end{aligned} \tag{3.12}$$

We define

$$L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \frac{\mu_1\varepsilon}{2}\int_\Omega u^2 dx + \varepsilon\int_\Omega \nabla u_t \nabla u dx, \tag{3.13}$$

where $0 < \varepsilon < 1$ to be given later and

$$0 < \sigma < \min\left\{\frac{p-2}{2p}, \frac{1}{\rho+2} - \frac{1}{p}\right\}. \tag{3.14}$$

We take the derivative of (3.13) and use the first equation in (2.4), we obtain

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \frac{\varepsilon}{\rho+1}\int_\Omega u_t^{\rho+2}dx - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|_m^m \\ &\quad -\mu_2\varepsilon\int_\Omega uz(x,1,t)\,dx + \varepsilon\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s)\,dsdx \\ &\quad +\varepsilon\|\nabla u_t\|^2 + \varepsilon b\|u\|_p^p. \end{aligned}$$

Utilizing Young's and Hölder's inequalities, for $\delta,\eta>0$,

$$\left|\mu_2\varepsilon\int_\Omega uz(x,1,t)\,dx\right| \le |\mu_2|\,\varepsilon\left(\delta\int_\Omega u^2dx + \frac{1}{4\delta}\int_\Omega z^2(x,1,t)\,dx\right)$$

and

$$\begin{aligned} &\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,dsdx \\ =& \int_\Omega\int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(s)-\nabla u(t))\,dsdx + \int_0^t g(t-s)\,ds\,\|\nabla u(t)\|^2 \\ \ge& -\eta(g\circ\nabla u)(t) + \left(1-\frac{1}{4\eta}\right)\int_0^t g(s)\,ds\,\|\nabla u(t)\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} L'(t) &\ge (1-\sigma) H^{-\sigma}(t) H'(t) + \frac{\varepsilon}{\rho+1}\int_\Omega u_t^{\rho+2}dx - \varepsilon\|\nabla u\|_m^m - \varepsilon\eta(g\circ\nabla u)(t) \\ &\quad +\varepsilon\left(-1-\left(\frac{1}{4\eta}-1\right)\int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \varepsilon\|\nabla u_t\|^2 \\ &\quad -|\mu_2|\,\varepsilon\left(\delta\|u\|^2 + \frac{1}{4\delta}\int_\Omega z^2(x,1,t)\,dx\right) + \varepsilon b\|u\|_p^p \\ &\ge \left[(1-\sigma)H^{-\sigma}(t) - \frac{\varepsilon|\mu_2|}{4\delta\alpha}\right]H'(t) + \frac{\varepsilon}{\rho+1}\int_\Omega u_t^{\rho+2}dx - \varepsilon\|\nabla u\|_m^m \\ &\quad +\varepsilon\left(-1-\left(\frac{1}{4\eta}-1\right)\int_0^t g(s)\,ds\right)\|\nabla u\|^2 - \varepsilon\eta(g\circ\nabla u)(t) \\ &\quad -|\mu_2|\,\varepsilon\delta\|u\|^2 + \varepsilon\|\nabla u_t\|^2 + \varepsilon b\|u\|_p^p, \end{aligned} \tag{3.15}$$

where $-\int_\Omega z^2(x,1,t)\,dx \ge -\frac{1}{\alpha}H'(t)$ holds by Lemma 3.1. (3.15) remains valid even if δ is time dependent since the integral is taken over the x-variable, hence, taking $\delta = \frac{|\mu_2|}{4\alpha k}H^\sigma(t)$, for large k to be specified later, we have

$$\begin{aligned} L'(t) &\ge (1-\sigma-\varepsilon k) H^{-\sigma}(t) H'(t) + \frac{\varepsilon}{\rho+1}\int_\Omega u_t^{\rho+2}dx - \varepsilon\|\nabla u\|_m^m \\ &\quad +\varepsilon\left(-1-\left(\frac{1}{4\eta}-1\right)\int_0^t g(s)\,ds\right)\|\nabla u\|^2 - \varepsilon\eta(g\circ\nabla u)(t) \\ &\quad -\frac{|\mu_2|^2\varepsilon}{4k\alpha}H^\sigma\|u\|^2 + \varepsilon\|\nabla u_t\|^2 + \varepsilon b\|u\|_p^p. \end{aligned}$$

Recalling the definition of $E(t)$ by (3.1) and adding $p(H(t) - E_2 + E(t))$, we arrive

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon k) H^{-\sigma}(t) H'(t) + \varepsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\int_\Omega u_t^{\rho+2}dx + \varepsilon\left(\frac{p}{m}-1\right)\|\nabla u\|_m^m \\ & +\varepsilon\left(\frac{p}{2}-\eta\right)(g\circ\nabla u)(t) + \varepsilon\left(\frac{p-2}{2}-\left(\frac{p-2}{2}+\frac{1}{4\eta}\right)\int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\ & -\frac{|\mu_2|^2\varepsilon}{4k\alpha}H^\sigma\|u\|^2 + \varepsilon p H(t) + \frac{\varepsilon\xi p}{2}\int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx - \varepsilon p E_2 \\ & +\frac{(p+2)\varepsilon}{2}\|\nabla u_t\|^2. \end{aligned} \tag{3.16}$$

Now, taking η to satisfy

$$\frac{1-l}{2(1-\beta)l(p-2)} < \eta < \frac{p(1-\beta)}{2}+\beta,$$

which derives from (3.3). Then, employing $l\|\nabla u\|^2 + (g\circ\nabla u)(t) \geq \lambda_2^2$ from (3.8), to obtain

$$\begin{aligned} & \left(\frac{p-2}{2}-\left(\frac{p-2}{2}+\frac{1}{4\eta}\right)\int_0^t g(s)\,ds\right)\|\nabla u\|^2 + \left(\frac{p}{2}-\eta\right)(g\circ\nabla u)(t) - pE_2 \\ \geq\ & \frac{\beta(p-2)}{2}\left(l\|\nabla u\|^2 + (g\circ\nabla u)(t)\right) - pE_2 \\ =\ & \frac{\beta(p-2)}{2}\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}\left(l\|\nabla u\|^2 + (g\circ\nabla u)(t)\right) \\ & +\frac{\beta(p-2)}{2}\frac{\lambda_1^2}{\lambda_2^2}\left(l\|\nabla u\|^2 + (g\circ\nabla u)(t)\right) - pE_2 \\ \geq\ & c_3\left(l\|\nabla u\|^2 + (g\circ\nabla u)(t)\right) + c_4, \end{aligned}$$

where $c_3 = \frac{\beta(p-2)}{2}\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2} > 0$ and $c_4 = \frac{\beta(p-2)}{2}\lambda_1^2 - pE_2$. Moreover, by $E_2 < \beta E_1$ and $E_1 = \frac{(p-2)}{2p}\lambda_1^2$, we have

$$c_4 = \frac{\beta(p-2)}{2}\lambda_1^2 - pE_2 > \beta\left(\frac{(p-2)\lambda_1^2}{2} - pE_1\right) = 0.$$

Thus, (3.16) becomes

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon k) H^{-\sigma}(t) H'(t) + \varepsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\int_\Omega u_t^{\rho+2}dx + \varepsilon\left(\frac{p}{m}-1\right)\|\nabla u\|_m^m \\ & +\varepsilon c_3\left(l\|\nabla u\|^2 + (g\circ\nabla u)(t)\right) - \frac{|\mu_2|^2\varepsilon}{4k\alpha}H^\sigma\|u\|^2 \\ & +\frac{\varepsilon\xi p}{2}\int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx + \frac{(p+2)\varepsilon}{2}\|\nabla u_t\|^2 + p\varepsilon H(t). \end{aligned} \tag{3.17}$$

By using (3.12), we arrive at $H^\sigma(t) \leq \left(\frac{b}{p}\right)^\sigma \|u\|_p^{\sigma p}$, hence

$$H^\sigma\|u\|_2^2 \leq \left(\frac{b}{p}\right)^\sigma |\Omega|^{\frac{p-2}{p}}\|u\|_p^{2+\sigma p}. \tag{3.18}$$

Substituting (3.18) into (3.17), letting $a_1 = \min\left\{c_3, \frac{p}{2}\right\}$, decomposing $\varepsilon pH(t)$ by

$$\varepsilon pH(t) = \varepsilon(2a_1 + (p - 2a_1))H(t),$$

and from (2.3), we conclude that

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon k) H^{-\sigma}(t) H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p-2a_1}{\rho+2}\right)\int_\Omega u_t^{\rho+2}dx \\ & +\varepsilon\left(\frac{p-2a_1}{m} - 1\right)\|\nabla u\|_m^m + \varepsilon(c_3 l - a_1 l)\|\nabla u\|^2 \\ & +\varepsilon(c_3 - a_1)(g\circ\nabla u)(t) + \varepsilon\frac{2a_1 b}{p}\|u\|_p^p - \frac{|\mu_2|^2\varepsilon}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}}\|u\|_p^{2+\sigma p} \\ & +\varepsilon\xi\left(\frac{p}{2} - a_1\right)\int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx \\ & +\varepsilon\left(\frac{p+2}{2} - a_1\right)\|\nabla u_t\|^2 + \varepsilon(p-2a_1)H(t). \end{aligned}$$

Then, for $2 + \sigma p \leq p$, utilizing Lemma 2.2, we obtain

$$\begin{aligned} L'(t) \geq\ & (1-\sigma-\varepsilon k) H^{-\sigma}(t) H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p-2a_1}{\rho+2}\right)\int_\Omega u_t^{\rho+2}dx \\ & +\varepsilon\left(\frac{p-2a_1}{m} - 1 - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}}\right)\|\nabla u\|_m^m \\ & +\varepsilon\left(c_3 l - a_1 l - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}}\right)\|\nabla u\|^2 \\ & +\varepsilon(c_3 - a_1)(g\circ\nabla u)(t) + \varepsilon\left(\frac{2a_1 b}{p} - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}}\right)\|u\|_p^p \\ & +\varepsilon\left(\frac{p+2}{2} - a_1\right)\|\nabla u_t\|^2 + \varepsilon\xi\left(\frac{p}{2} - a_1\right)\int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx \\ & +\varepsilon(p-2a_1)H(t). \end{aligned} \tag{3.19}$$

Here, choosing the constant k large enough, such that

$$c_3 l - a_1 l - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}} > 0,\ \frac{2a_1 b}{p} - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}} > 0$$

and

$$\frac{p-2a_1}{m} - 1 - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^\sigma|\Omega|^{\frac{p-2}{p}} > 0.$$

Choosing ε small enough, such that

$$1 - \sigma - \varepsilon k > 0$$

and

$$L(0) = H^{1-\sigma}(0) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_1|^\rho u_1 u_0 dx + \frac{\mu_1\varepsilon}{2}\int_\Omega u_0^2 dx + \varepsilon\int_\Omega \nabla u_1\cdot\nabla u_0 dx > 0. \tag{3.20}$$

Therefore, there exists $K > 0$, such that

$$\begin{aligned} L'(t) \geq\ & \varepsilon K\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|\nabla u\|_m^m + (g\circ\nabla u)(t) + \|\nabla u_t\|^2 \\ & + H(t) + \|u\|_p^p + \int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx\Big), \end{aligned} \tag{3.21}$$

which together with (3.20) implies that

$$L(t) \geq L(0) > 0, \text{ for } t \geq 0.$$

Otherwise, utilizing Young's and Hölder's inequalities, we get

$$\begin{aligned} \left|\int_\Omega |u_t|^\rho u_t u dx\right|^{\frac{1}{1-\sigma}} &\leq \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}}\|u\|_{\rho+2}^{\frac{1}{1-\sigma}} \leq c_5\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}}\|u\|_\rho^{\frac{1}{1-\sigma}} \\ &\leq c_6\left(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\mu} + \|u\|_\rho^{\frac{1}{1-\sigma}\theta}\right), \end{aligned} \tag{3.22}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$ and $c_5, c_6 > 0$. Choosing $\mu = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1$, then from (3.14), we see that $\frac{\theta}{1-\sigma} = \frac{\rho+2}{(1-\sigma)(\rho+2)-(\rho+1)} < p$. Hence, from Lemma 2.2 and (3.22), we obtain

$$\left|\int_\Omega |u_t|^\rho u_t u dx\right|^{\frac{1}{1-\sigma}} \leq c_7\left(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|\nabla u\|^2 + \|u\|_p^p\right), \tag{3.23}$$

with $c_7 > 0$. In a similar way, as in deriving (3.22), we also get

$$\left|\int_\Omega |\nabla u_t \nabla u|\,dx\right|^{\frac{1}{1-\sigma}} \leq c_8\left(\|\nabla u_t\|^2 + \|\nabla u\|_2^{\frac{2}{1-2\sigma}}\right), \tag{3.24}$$

for $c_8 > 0$. Combining (3.13), (3.23) and (3.24) to satisfy

$$\begin{aligned} L(t)^{\frac{1}{1-\sigma}} &= \left(H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \frac{\mu_1\varepsilon}{2}\|u\|^2 + \varepsilon\int_\Omega \nabla u_t\nabla u dx\right)^{\frac{1}{1-\sigma}} \\ &\leq c_9\Big(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|u\|_p^p + \|\nabla u\|^2 \\ &\quad + \|u\|_2^{\frac{2}{1-\sigma}} + \|\nabla u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla u_t\|^2\Big) \\ &\leq c_{10}\Big(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|u\|_p^p + \|\nabla u\|^2 \\ &\quad + \|u\|_p^{\frac{2}{1-\sigma}} + \|\nabla u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla u_t\|^2\Big), \end{aligned} \tag{3.25}$$

for $t \geq 0$ and c_9, $c_{10} > 0$. By using (3.10) and (3.11), such that

$$\|\nabla u\|_2^{\frac{2}{1-2\sigma}} \leq K_1^{\frac{2}{1-2\sigma}} \leq K_1^{\frac{2}{1-2\sigma}} \frac{H(t)}{H(0)} \text{ and } \|u\|_p^{\frac{2}{1-\sigma}} \leq K_1^{\frac{2}{1-2\sigma}} \frac{H(t)}{H(0)}. \tag{3.26}$$

From (3.25) and (3.26), we get

$$L(t)^{\frac{1}{1-\sigma}} \leq c_{11} \left(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|u\|_p^p + \|\nabla u\|^2 + \|\nabla u_t\|^2 \right), \ t \geq 0, \tag{3.27}$$

with $c_{11} > 0$. Combining (3.27) with (3.21), we obtain

$$L'(t) \geq c_{12} L(t)^{\frac{1}{1-\sigma}}, \ t \geq 0, \tag{3.28}$$

here $c_{12} = \frac{\varepsilon K}{c_{11}}$. A simple integration of (3.28) over $(0, t)$, we have

$$L(t) \geq \left(L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{\sigma c_{12}}{1-\sigma} t \right)^{-\frac{1-\sigma}{\sigma}}. \tag{3.29}$$

As we know, $L(0) > 0$, (3.29) indicates that L becomes infinite in a finite time T with $0 < T \leq \frac{1-\sigma}{c_{12}\sigma L(0)^{\frac{\sigma}{1-\sigma}}}$. As a result, we completed the proof. Q.E.D.

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References

[1] M. Kafini and S.A. Messaoudi, *A blow-up result in a nonlinear wave equation with delay*, Mediterranean Journal of Mathematics, **13** (2016), 237–247.

[2] R. Datko, J. Lagnese and M.P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM Journal on Control and Optimization, **24**(1) (1986), 152-156.

[3] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM Journal on Control and Optimization, **45**(5) (2006), 1561-1585.

[4] S.T. Wu, *Blow-up of solution for a viscoelastic wave equation with delay*, Acta Mathematica Scientia, **39**B(1) (2019), 329-338.

[5] W.J. Liu, *General decay and blow-up of solution for a quasilinear viscoelastic problem with nonlinear source*, Nonlinear Analysis, Theory, Methods & Applications, **73** (2010), 1890–1904.

[6] M.M. Cavalcanti, V.N. Domingos Cavalcanti and J.A. Soriano, *Exponential decay for the solution of semilinear viscoelastic wave equation with localized damping*, Electronic Journal of Differential Equations, **44** (2002), 1-14.

[7] X. Han and M. Wang, *Global existence and uniform decay for a non-linear viscoelastic equation with damping*, Nonlinear Analysis, **70** (2009), 3090-3098.

[8] W.J. Liu, G. Li and L.H. Hong, *General decay and blow-up of solutions for a system of viscoelastic equations of Kirchhoff type with strong damping*, Journal of Function Spaces, 2014, (2014), 1-21, Article ID 284809.

[9] W.J. Liu and J. Yu, *On decay and blow-up of the solution for a viscoelastic wave equation with boundary damping and source terms*, Nonlinear Analysis, Theory, Methods & Applications, **74**(6) (2011), 2175–2190.

[10] W.J. Liu, Y. Sun and G. Li, *On decay and blow-up of solutions for a singular nonlocal viscoelastic problem with a nonlinear source term*, Topological Methods in Nonlinear Analysis, **49**(1) (2017), 299–323.

[11] N. Mezouar and S. Boulaaras, *Global existence and decay of solutions for a class of viscoelastic Kirchhoff equation*, Bulletin of the Malaysian Mathematical Sciences Society, **43** (2020), 725–755.

[12] N. Mezouar and S. Boulaaras, *Global existence of solutions to a viscoelastic non-degenerate Kirchhoff equation*, Applicable Analysis, **99**(10) 2020, 1724–1748.

[13] M.I. Mustafa, *General decay result for nonlinear viscoelastic equations*, Journal Applied Mathematics, **457** (2018), 134-152.

[14] E. Pişkin and H. Yüksekkaya, *Local existence and blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay*, Computational Methods for Differential Equations, **9(2)** (2021), 623–636.

[15] R. Abazari, K. Yıldırım, *Numerical study of Sivashinsky equation using a splitting scheme based on Crank-Nicolson method*, Mathematical Methods in the Applied Sciences, **42** (2019), 5509–5521.

[16] M. Bayram, K. Yıldırım, *Approximates method for solving an elasticity problem of settled of the elastic ground with variable coefficients*, Applied Mathematics and Information Sciences, **7**(4) (2013), 1351-1357.

[17] K. Yıldırım, B. İbiş, M. Bayram, *New solutions of the nonlinear Fisher type equations by the reduced differential transform*, Nonlinear Science Letters A, **3**(1) (2012), 29-36.

[18] R.A. Adams, J.J.F. Fournier, *Sobolev Spaces*, Academic Press, 2003.

[19] E. Pişkin, *Sobolev Spaces*, Seçkin Publishing, 2017. (in Turkish)

[20] S. Nicaise and C. Pignotti, *Stabilization of the wave equation with boundary or internal distributed delay*, Differential and Integral Equations, **21**(9-10) (2008), 935-958.

[21] M. Kirane and S.H. Belkacem, *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Zeitschrift für angewandte Mathematik und Physik, **62** (2011), 1065–1082.

[22] S.T. Wu, *Asymptotic behavior for a viscoelastic wave equation with a delay term*, Taiwanese Journal of Mathematics, **17** (2013), 765–784.

[23] M.M. Cavalcanti, V.N. Domingos Cavalcanti and J. Ferreira, *Existence and uniform decay of nonlinear viscoelastic equation with strong damping*, Mathematical Methods in the Applied Sciences, **24** (2001), 1043–1053.